Second eigenvalue of a Jacobi operator of hypersurfaces with constant scalar curvature

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Abstract

Let $x : M \to \mathbb{S}^{n+1}(1)$ be an n-dimensional compact hypersurface with constant scalar curvature $n(n-1)r,\ r \geq 1$, in a unit sphere $\mathbb{S}^{n+1}(1),\ n \geq 5$. We know that such hypersurfaces can be characterized as critical points for a variational problem of the integral $\int_M H dv$ of the mean curvature H. In this paper, we derive an optimal upper bound for the second eigenvalue of the Jacobi operator J_s of M. Moreover, when $r > 1$, the bound is attained if and only if M is totally umbilical and non-totally geodesic, when $r = 1$, the bound is attained if M is the Riemannian product $\mathbb{S}^m(c) \times \mathbb{S}^{n-m}(\sqrt{1-c^2}),\ 1 \leq m \leq n-2,\ c = \sqrt{\frac{(n-1)m+\sqrt{(n-1)m(n-m)}}{n(n-1)}}$.

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1 Introduction

Let M be an n-dimensional compact hypersurface in a unit sphere $\mathbb{S}^{n+1}(1)$. We denote the components of the second fundamental form of M by h_{ij}, and denote the principal curvatures of M by $k_1, \ldots, k_n$. Let H, H_2 and H_3 denote the mean curvature, the 2nd mean curvature and the 3rd mean curvature of M respectively, namely,

$$H = \frac{1}{n}\sum_{i=1}^{n} k_i,\ H_2 = \frac{2}{n(n-1)}\sum_{1\leq i_1<i_2\leq n} k_{i_1}k_{i_2},$$
$$H_3 = \frac{6}{n(n-1)(n-2)}\sum_{1\leq i_1<i_2<i_3\leq n} k_{i_1}k_{i_2}k_{i_3}.$$

We denote the square norm of the second fundamental form of M by S. The Schrödinger operator $J_m = -\Delta - S - n$, where Δ stands for the Laplace-Beltrami operator, is called the Jacobi operator. Its spectral behavior is directly related to the instability of both the minimal hypersurfaces and the hypersurfaces with constant mean curvature in $\mathbb{S}^{n+1}(1)$ (cf. [19] and [3]). The first eigenvalue of the Jacobi operator J_m of such hypersurfaces in $\mathbb{S}^{n+1}(1)$ was studied by Simons [19] and Wu [22].

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The second eigenvalue of the Jacobi operator J_m of the compact hypersurfaces in $\mathbb{S}^{n+1}(1)$ was studied by A. El Soufi and S. Ilias in [20]. They obtained that if M is an n-dimensional compact hypersurface in $\mathbb{S}^{n+1}(1)$, then the second eigenvalue $\lambda_2^{J_m}$ of the Jacobi operator J_m satisfies

$$\lambda_2^{J_m} \leq 0,$$

where the equality holds if and only if M is a totally umbilical hypersurface in $\mathbb{S}^{n+1}(1)$.

For any C^2-function f on M, we define a differential operator

$$\Box f = \sum_{i,j=1}^{n} (nH\delta_{ij} - h_{ij}) f_{ij}, \tag{1.1}$$

where (f_{ij}) is the Hessian of f. The differential operator $\Box$ is self-adjoint and it was introduced by S. Y. Cheng and Yau in [8] in order to study the compact hypersurfaces with constant scalar curvature in $\mathbb{S}^{n+1}(1)$. They proved that if M is an n-dimensional compact hypersurface with constant scalar curvature $n(n-1)r,\ r \geq 1$, and if the sectional curvature of M is non-negative, then M is either a totally umbilical hypersurface $\mathbb{S}^n(c)$ or a Riemannian product $\mathbb{S}^m(c) \times \mathbb{S}^{n-m}(\sqrt{1-c^2}),\ 1 \leq m \leq n-1$, where $\mathbb{S}^k(c)$ denotes a sphere of radius c. In [12], the first author proved that if M is an n-dimensional ($n \geq 3$) compact hypersurface with constant scalar curvature $n(n-1)r,\ r \geq 1$, and if $S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$, then M is either a totally umbilical hypersurface or a Riemannian product $\mathbb{S}^1(c) \times \mathbb{S}^{n-1}(\sqrt{1-c^2})$ with $0 < 1-c^2 = \frac{n-2}{nr} \leq \frac{n-2}{n}$. Furthermore, the Riemannian product $\mathbb{S}^1(c) \times \mathbb{S}^{n-1}(\sqrt{1-c^2})$ has been characterized in [5] and [6].

In [1], Alencar, do Carmo and Colares studied the stability of the hypersurfaces with constant scalar curvature in $\mathbb{S}^{n+1}(1)$. In this case, the Jacobi operator J_s is given by (cf. [1] and [7])

$$J_s = -\Box - \{n(n-1)H + nHS - f_3\}, \tag{1.2}$$

which is associated with the variational characterization of the hypersurfaces with constant scalar curvature in $\mathbb{S}^{n+1}(1)$, where $f_3 = \sum_{j=1}^{n} k_j^3$ (cf. [17] and [18]). The spectral behavior of J_s is directly related to the instability of the hypersurfaces with constant scalar curvature.

In general, J_s is not an elliptic operator. When $r > 1,\ n^2H^2 > S > 0$, the differential operator $\Box$ and hence J_s is an elliptic operator (cf. pages 3310, 3311 in [7]). When $r = 1$, if we assume that $H_3 \neq 0$ on M, then we have $H \neq 0$ and J_s is elliptic (cf. Proposition 1.5 in [11]).

Definition 1: We call $\lambda_i^{J_s}$ an eigenvalue of J_s if there exists a non-zero function f on M such that $J_s f = \lambda_i^{J_s} f$, we call $\lambda_i^{\Box}$ an eigenvalue of $\Box$ if there exists a non-zero function f on M such that $\Box f + \lambda_i^{\Box} f = 0$, and we call λ_i^{Δ} an eigenvalue of Δ if there exists a non-zero function f on M such that $\Delta f + \lambda_i^{\Delta} f = 0$.

In [7], Q. -M. Cheng studied the first eigenvalue of J_s of the hypersurfaces with constant scalar curvature $n(n-1)r, r > 1$ in $\mathbb{S}^{n+1}(1)$, and derived an optimal upper bound for the first eigenvalue of J_s.

Theorem 1.1. *(see Corollary 1.2 in [7]) Let M be an n-dimensional compact orientable hypersurface with constant scalar curvature $n(n-1)r,\ r > 1$, in $\mathbb{S}^{n+1}(1)$. Then the Jacobi operator J_s is elliptic and the first eigenvalue of J_s satisfies*

$$\lambda_1^{J_s} \leq -n(n-1)r\sqrt{r-1},$$

where the equality holds if and only if M is totally umbilical and non-totally geodesic.

In [2], L. J. Alías, A. Brasil and L. A. M. Sousa studied the first eigenvalue $\lambda_1^{J_s}$ of J_s of the hypersurfaces with constant scalar curvature $n(n-1)$ in $\mathbb{S}^{n+1}(1)$.

Theorem 1.2. *(see Theorem 2 in [2]) Let M be an n-dimensional compact orientable hypersurface with constant scalar curvature $n(n-1)$, in $\mathbb{S}^{n+1}(1)$, $n \geq 3$. Assume that $H_3 \neq 0$, then the Jacobi operator J_s is elliptic and the first eigenvalue $\lambda_1^{J_s}$ of the Jacobi operator J_s satisfies*

$$\lambda_1^{J_s} \leq -2n(n-1) \min |H|,$$

where the equality holds if and only if M is the Riemannian product $\mathbb{S}^m(c) \times \mathbb{S}^{n-m}(\sqrt{1-c^2})$ with $1 \leq m \leq n-2$, $c = \sqrt{\frac{(n-1)m+\sqrt{(n-1)m(n-m)}}{n(n-1)}}$.

In this paper, we study the second eigenvalue for J_s of the hypersurfaces with constant scalar curvature $n(n-1)r, r \geq 1$ in $\mathbb{S}^{n+1}(1)$, $n \geq 5$, and we have the following results.

Theorem 1.3. *Let M be an n-dimensional compact orientable hypersurface with constant scalar curvature $n(n-1)r$, $r > 1$, in $\mathbb{S}^{n+1}(1)$, $n \geq 5$. Then, the Jacobi operator J_s is elliptic and the second eigenvalue $\lambda_2^{J_s}$ of the Jacobi operator J_s satisfies*

$$\lambda_2^{J_s} \leq 0,$$

where the equality holds if and only if M is totally umbilical and non-totally geodesic.

Theorem 1.4. *Let M be an n-dimensional compact orientable hypersurface with constant scalar curvature $n(n-1)$, in $\mathbb{S}^{n+1}(1)$, $n \geq 5$. Assume that $H_3 \neq 0$, then the Jacobi operator J_s is elliptic and the second eigenvalue $\lambda_2^{J_s}$ of the Jacobi operator J_s satisfies*

$$\lambda_2^{J_s} \leq -\frac{n(n-1)(n-2)}{2} \min |H_3|, \tag{1.3}$$

where the equality holds if and only if $H_3 = constant \neq 0$ and the position functions of M in $\mathbb{S}^{n+1}(1)$ are the second eigenfunctions of J_s corresponding to $\lambda_2^{J_s}$. In particular, when M is the Riemannian product $\mathbb{S}^m(c) \times \mathbb{S}^{n-m}(\sqrt{1-c^2})$, $1 \leq m \leq n-2$, $c = \sqrt{\frac{(n-1)m+\sqrt{(n-1)m(n-m)}}{n(n-1)}}$, the equality in (1.3) is attained.

2 Preliminaries

Throughout this paper, all manifolds are assumed to be smooth and connected without boundary. Let $x : M \to \mathbb{S}^{n+1}(1)$ be an n-dimensional hypersurface in a unit sphere $\mathbb{S}^{n+1}(1)$. We make the following convention on the range of indices:

$$1 \leq i, j, k, l \leq n.$$

Let $\{e_1, \cdots, e_n, e_{n+1}\}$ be a local orthonormal frame with dual coframe $\{\omega_1, \cdots, \omega_n, \omega_{n+1}\}$ such that when restricted on M, $\{e_1, \cdots, e_n\}$ is a local orthonormal frame on M. Hence we have $\omega_{n+1} = 0$ on M and we have the following structure equations (see [4], [9], [12] and [19]):

$$dx = \sum_i \omega_i e_i, \tag{2.1}$$

$$de_i = \sum_j \omega_{ij} e_j + \sum_j h_{ij} \omega_j e_{n+1} - \omega_i x, \tag{2.2}$$

$$de_{n+1} = -\sum_{i,j} h_{ij}\omega_j e_i, \tag{2.3}$$

where h_{ij} denote the components of the second fundamental form of M.

The Gauss equations are (see [9], [12])

$$R_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + h_{ik}h_{jl} - h_{il}h_{jk}, \tag{2.4}$$

$$R_{ik} = (n-1)\delta_{ik} + nHh_{ik} - \sum_j h_{ij}h_{jk}, \tag{2.5}$$

$$R = n(n-1)r = n(n-1) + n^2H^2 - S, \tag{2.6}$$

where R is the scalar curvature of M, r is the normalized scalar curvature of M and $S = \sum_{i,j} h_{ij}^2$ is the norm square of the second fundamental form, $H = \frac{1}{n}\sum_i h_{ii}$ is the mean curvature of M.

The Codazzi equations are given by (see [9], [12])

$$h_{ijk} = h_{ikj}. \tag{2.7}$$

Let f be a smooth function on M, we define its gradient and Hessian by (see [9], [12])

$$df = \sum_{i=1}^{n} f_i\omega_i, \tag{2.8}$$

$$\sum_{j=1}^{n} f_{ij}\omega_j = df_i + \sum_{j=1}^{n} f_j\omega_{ji}. \tag{2.9}$$

Then the Jacobi operator J_s (see (1.2)) is defined by

$$\begin{aligned} J_s f &= -\Box f - \{n(n-1)H + nHS - f_3\}f \\ &= -\sum_{i,j}(nH\delta_{ij} - h_{ij})f_{ij} - \{n(n-1)H + nHS - f_3\}f. \end{aligned} \tag{2.10}$$

3 Some examples and some lemmas

First of all, we consider the first and second eigenvalues of the Jacobi operator J_s of the totally umbilical and non-totally geodesic hypersurface in $\mathbb{S}^{n+1}(1)$ with constant scalar curvature $n(n-1)r,\ r > 1$ and the Riemannian product $\mathbb{S}^m(c) \times \mathbb{S}^{n-m}(\sqrt{1-c^2}),\ 1 \le m \le n-2$ with constant scalar curvature $n(n-1)$ in $\mathbb{S}^{n+1}(1),\ n \ge 3$.

Example 3.1. Let M be a totally umbilical and non-totally geodesic hypersurface with constant scalar curvature $n(n-1)r,\ r > 1$ in $\mathbb{S}^{n+1}(1)$. We can assume $H > 0$. In this case, $\Box = (n-1)H\Delta$, from $S = nH^2$ and the Gauss equation (2.6) we have $H = \sqrt{r-1}$. By (1.2) we have

$$J_s = -\Box - \{n(n-1)H + nHS - f_3\} = -\{(n-1)H\Delta + n(n-1)H(1+H^2)\},$$

hence the eigenvalues $\lambda_i^{J_s}$ of J_s are given by

$$\lambda_i^{J_s} = (n-1)H\lambda_i^{\Delta} - n(n-1)H(1+H^2),$$

where λ_i^{Δ} denotes the eigenvalue of Δ (see Definition 1). It is well-known that $\lambda_1^{\Delta} = 0,\ \lambda_2^{\Delta} = nr = n(1+H^2)$, hence we have

$$\begin{aligned} \lambda_1^{J_s} &= -n(n-1)H(1+H^2) = -n(n-1)r\sqrt{r-1} < 0, \\ \lambda_2^{J_s} &= (n-1)H \cdot n(1+H^2) - n(n-1)H(1+H^2) = 0. \end{aligned} \tag{3.1}$$

Example 3.2. Let M be the Riemannian product

$$\mathbb{S}^m(c) \times \mathbb{S}^{n-m}(\sqrt{1-c^2}),\ 1 \leq m \leq n-2,\ c = \sqrt{\frac{(n-1)m + \sqrt{(n-1)m(n-m)}}{n(n-1)}}$$

in $\mathbb{S}^{n+1}(1),\ n \geq 3$. In this case, the position vector is

$$x = (x_1, x_2) \in \mathbb{S}^m(c) \times \mathbb{S}^{n-m}(\sqrt{1-c^2})$$

and the unit normal vector at this point x is given by $e_{n+1} = (\frac{\sqrt{1-c^2}}{c}x_1, -\frac{c}{\sqrt{1-c^2}}x_2)$.

Its principal curvatures are given by

$$k_1 = \cdots = k_m = -\frac{\sqrt{1-c^2}}{c},\ k_{m+1} = \cdots = k_n = \frac{c}{\sqrt{1-c^2}}. \tag{3.2}$$

Since the principal curvatures are constant hence $H,\ S,\ f_3$ are all constant given by

$$\begin{aligned} H &= \frac{nc^2 - m}{cn\sqrt{1-c^2}}, \\ S &= \frac{m(1-c^2)}{c^2} + \frac{(n-m)c^2}{1-c^2} = n^2H^2, \\ f_3 &= -\frac{m(1-c^2)^{3/2}}{c^3} + \frac{(n-m)c^3}{(1-c^2)^{3/2}}. \end{aligned} \tag{3.3}$$

After a long but straightforward computation, we know that M has constant scalar curvature $n(n-1)$ and

$$H_3 = -\frac{2H}{n-2} = -\frac{2(nc^2-m)}{cn(n-2)\sqrt{1-c^2}} < 0, \tag{3.4}$$

hence the Jacobi operator J_s is elliptic (cf. Proposition 1.5 in [11]). We also have

$$n(n-1)H + nHS - f_3 = \frac{(n-2m)(n-1)c^4 + 2m(m-1)c^2 - m(m-1)}{c^3(1-c^2)^{3/2}}, \tag{3.5}$$

thus the Jacobi operator $J_s = -\square - \{n(n-1)H + nHS - f_3\}$ becomes

$$J_s = -\square - \frac{(n-2m)(n-1)c^4 + 2m(m-1)c^2 - m(m-1)}{c^3(1-c^2)^{3/2}}, \tag{3.6}$$

hence, the eigenvalues $\lambda_i^{J_s}$ of J_s are given by

$$\lambda_i^{J_s} = \lambda_i^{\square} - \frac{(n-2m)(n-1)c^4 + 2m(m-1)c^2 - m(m-1)}{c^3(1-c^2)^{3/2}}, \tag{3.7}$$

where $\lambda_i^{\square}$ denotes the eigenvalue of the differential operator $\square$ (see Definition 1).

Since the differential operator $\square$ is self-adjoint and M is compact, we have $\lambda_1^{\square} = 0$ and its corresponding eigenfunctions are non-zero constant functions, hence

$$\lambda_1^{J_s} = -\frac{(n-2m)(n-1)c^4 + 2m(m-1)c^2 - m(m-1)}{c^3(1-c^2)^{3/2}}. \tag{3.8}$$

Let $\{e_1, \cdots, e_n\}$ be a local orthonormal basis of TM with dual basis $\{\omega_1, \cdots, \omega_n\}$ such that $\{e_1, \cdots, e_m\}$ is a local orthonormal basis of $T\mathbb{S}^m(c)$ when restricted on $\mathbb{S}^m(c)$ and $\{e_{m+1}, \cdots, e_n\}$ is a local orthonormal basis of $T\mathbb{S}^{n-m}(\sqrt{1-c^2})$ when restricted on $\mathbb{S}^{n-m}(\sqrt{1-c^2})$. So we have

$$\square f = \sum_{i=1}^{m}(nH-k_1)f_{ii} + \sum_{j=m+1}^{n}(nH-k_n)f_{jj} = (nH-k_1)\Delta_1 f + (nH-k_n)\Delta_2 f, \tag{3.9}$$

where Δ_1 and Δ_2 denote the Laplace-Beltrami operators on $\mathbb{S}^m(c)$ and $\mathbb{S}^{n-m}(\sqrt{1-c^2})$ respectively. Since $(nH-k_1) = \frac{(n-1)c^2-(m-1)}{c\sqrt{1-c^2}} > 0$, $(nH-k_n) = \frac{(n-1)c^2-m}{c\sqrt{1-c^2}} > 0$, we conclude that

$$\lambda_2^{\square} = \min\{(nH-k_1)\lambda_2^{\Delta_1}, (nH-k_n)\lambda_2^{\Delta_2}\}, \tag{3.10}$$

where $\lambda_2^{\Delta_1}$ and $\lambda_2^{\Delta_2}$ are the second eigenvalues (or the first non-zero eigenvalue) of Δ_1 and Δ_2 which are given by

$$\lambda_2^{\Delta_1} = \frac{m}{c^2},\ \lambda_2^{\Delta_2} = \frac{n-m}{1-c^2}. \tag{3.11}$$

Therefore, from (3.10) and (3.11), after a direct computation, we have

$$\begin{aligned}\lambda_2^{J_s} &= \min\{(nH-k_1)\frac{m}{c^2} - \frac{(n-2m)(n-1)c^4+2m(m-1)c^2-m(m-1)}{c^3(1-c^2)^{3/2}},\\ &\quad (nH-k_n)\frac{n-m}{1-c^2} - \frac{(n-2m)(n-1)c^4+2m(m-1)c^2-m(m-1)}{c^3(1-c^2)^{3/2}}\}\\ &= \min\{\frac{(n-m)[(1-n)c^2+m]}{c(1-c^2)^{3/2}}, \frac{-m[(n-1)c^2-(m-1)]}{c^3(1-c^2)^{1/2}}\}.\end{aligned} \tag{3.12}$$

Since $c = \sqrt{\frac{(n-1)m+\sqrt{(n-1)m(n-m)}}{n(n-1)}}$, we have

$$\begin{aligned}&\frac{(n-m)[(1-n)c^2+m]}{c(1-c^2)^{3/2}} - \frac{m[(n-1)c^2-(m-1)]}{c^3(1-c^2)^{1/2}}\\ &= -\frac{n(n-1)c^4+2m(1-n)c^2+m(m-1)}{c^3(1-c^2)^{3/2}} = 0.\end{aligned} \tag{3.13}$$

It follows from (3.12) and (3.13) that

$$\lambda_2^{J_s} = \frac{(n-m)[(1-n)c^2+m]}{c(1-c^2)^{3/2}} < 0. \tag{3.14}$$

On the other hand, we also have

$$\begin{aligned}&-\frac{(n-2m)(n-1)c^4+2m(m-1)c^2-m(m-1)}{c^3(1-c^2)^{3/2}} + 2n(n-1)H\\ &= -\frac{(2c^2-1)(n(n-1)c^4+2m(1-n)c^2+m(m-1))}{c^3(1-c^2)^{3/2}} = 0,\end{aligned} \tag{3.15}$$

$$\begin{aligned}&\frac{(n-m)[(1-n)c^2+m]}{c(1-c^2)^{3/2}} + n(n-1)H\\ &= -\frac{n(n-1)c^4+2m(1-n)c^2+m(m-1)}{c(1-c^2)^{3/2}} = 0,\end{aligned} \tag{3.16}$$

and

$$
\begin{aligned}
&\frac{(n-m)[(1-n)c^2+m]}{c(1-c^2)^{3/2}}-\frac{n(n-1)(n-2)}{2}H_3\\
&=-\frac{(n(n-1)c^4+2m(1-n)c^2+m(m-1))(c^2(2n-1)-2m+1)}{c^3(1-c^2)^{3/2}}=0,
\end{aligned}
\tag{3.17}
$$

hence, from (3.8), (3.14), (3.15), (3.16) and (3.17), we have

$$
\lambda_1^{J_s}=-2n(n-1)H<\lambda_2^{J_s}=-n(n-1)H=\frac{n(n-1)(n-2)}{2}H_3<0. \tag{3.18}
$$

In the following we will assume that $x: M\to\mathbb{S}^{n+1}(1)$ is an n-dimensional compact orientable hypersurface with constant scalar curvature $n(n-1)r,\ r\geq 1$, in $\mathbb{S}^{n+1}(1),\ n\geq 5$, when $r=1$, we assume moreover $H_3\neq 0$. When $r>1$, we have $n^2H^2>S>0$, when $r=1$, since $H_3\neq 0$, we have $H\neq 0$. Hence, we can assume $H>0$ (cf. [7] and [11]).

Let a be a fixed vector in $\mathbb{R}^{n+2}$. We define functions $f^a: M\to\mathbb{R}$ and $\tilde{g}^a: M\to\mathbb{R}$ by

$$
f^a=<a,x>,\ \tilde{g}^a=<a,e_{n+1}>, \tag{3.19}
$$

where x is the position vector and e_{n+1} is the unit normal vector.

By using the structure equations and the definition of the covariant derivatives, we have the following result.

Lemma 3.3. *(see [4]) The gradient and the second derivative of the functions f and $\tilde{g}$ are given by*

$$
\begin{aligned}
&f_i^a=<a,e_i>,\ f_{ij}^a=\tilde{g}^a h_{ij}-f^a\delta_{ij},\\
&\tilde{g}_j^a=-\sum_{i=1}^n<a,e_i>h_{ij},\ \tilde{g}_{jk}^a=-\sum_{i=1}^n<a,e_i>h_{ijk}-\sum_{i=1}^n\tilde{g}^a h_{ij}h_{ik}+f^a h_{jk}.
\end{aligned}
\tag{3.20}
$$

Proof. By (2.1) we have

$$
df^a=<a,dx>=\sum_i<a,e_i>\omega_i,
$$

thus from (2.8) we have

$$
f_i^a=<a,e_i>. \tag{3.21}
$$

From (2.2) and (3.21) we have

$$
\begin{aligned}
\sum_{j=1}^n f_{ij}^a\omega_j&=df_i+\sum_{j=1}^n f_j\omega_{ji}=<a,de_i>+\sum_{j=1}^n<a,e_j>\omega_{ji}\\
&=\sum_{j=1}^n<a,e_{n+1}>h_{ij}\omega_j-<a,x>\omega_i,
\end{aligned}
$$

hence we have

$$
f_{ij}^a=<a,e_{n+1}>h_{ij}-<a,x>\delta_{ij}=\tilde{g}^a h_{ij}-f^a\delta_{ij}. \tag{3.22}
$$

After an analogous argument, we have

$$
\tilde{g}_j^a=-\sum_{i=1}^n<a,e_i>h_{ij},\ \tilde{g}_{jk}^a=-\sum_{i=1}^n<a,e_i>h_{ijk}-\sum_{i=1}^n\tilde{g}^a h_{ij}h_{ik}+f^a h_{jk}. \tag{3.23}
$$

□

We will use a technique which was introduced by Li and Yau in [13] and was later used by other authors (see [14], [16] and [21]).

Let B^{n+2} be the open unit ball in $\mathbb{R}^{n+2}$. For each point $g \in B^{n+2}$, we consider the map

$$F_g(p) = \frac{p + (\mu < p, g > +\lambda)g}{\lambda(< p, g > +1)}, \ \forall\ p \in \mathbb{S}^{n+1}(1) \subset \mathbb{R}^{n+2}, \tag{3.24}$$

where $\lambda = (1 - \|g\|^2)^{-1/2}$, $\mu = (\lambda - 1)\|g\|^{-2}$ and $<,>$ denotes the usual inner product on $\mathbb{R}^{n+2}$. A direct computation (see [14], [21]) shows that F_g is a conformal transformation from $\mathbb{S}^{n+1}(1)$ to $\mathbb{S}^{n+1}(1)$ and the differential map dF_g of F_g is given by

$$dF_g(v) = \lambda^{-2}(< p, g > +1)^{-2}\{\lambda(< p, g > +1)v - \lambda < v, g > p+ < v, g > (1 - \lambda)\|g\|^{-2}g\},$$

where v is a tangent vector to $\mathbb{S}^{n+1}$ at the point p. Hence, for two vectors $v,\ w \in T_p\mathbb{S}^{n+1}$ we have (see [14], [16] and [21])

$$< dF_g(v), dF_g(w) >= \frac{1 - \|g\|^2}{(< p, g > +1)^2} < v, w > .$$

By use of the technique in Li-Yau [13], we have the following result:

Lemma 3.4. *(see [14], [16] and [21])*

Let $x : M \to \mathbb{S}^{n+1}$ be a compact hypersurface in $\mathbb{S}^{n+1}$ with constant scalar curvature $n(n-1)r,\ r \geq 1$, and u be a positive first eigenfunction of the Jacobi operator J_s on M, then there exists $g \in B^{n+2}$ such that $\int_M u(F_g \circ x)dv = (0, \ldots, 0)$.

Let $\{E^A\}_{A=1}^{n+2}$ be a fixed orthonormal basis of $\mathbb{R}^{n+2}$, for a fixed point $g \in B^{n+2}$, we define functions $f^A : M \to \mathbb{R}(1 \leq A \leq n+2)$ by

$$f^A =< E^A, F_g \circ x >= \frac{< E^A, x > +(\mu < x, g > +\lambda) < g, E^A >}{\lambda(< x, g > +1)}, \ \forall 1 \leq A \leq n+2. \tag{3.25}$$

Lemma 3.5. *The gradient of f^A is given by*

$$f_i^A = \frac{< E^A, e_i >}{\lambda(< x, g > +1)} + \frac{< g, e_i >}{\lambda(< x, g > +1)^2}(- < E^A, x > + \frac{1 - \lambda}{\lambda\|g\|^2} < g, E^A >). \tag{3.26}$$

Proof. By applying Lemma 3.3, we have

$$\begin{aligned}
f_i^A &= \frac{< E^A, e_i > +\mu < g, e_i >< g, E^A >}{\lambda(< x, g > +1)} - f^A\frac{< g, e_i >}{< x, g > +1} \\
&= \frac{< E^A, e_i >}{\lambda(< x, g > +1)} + \frac{< g, e_i >}{\lambda(< x, g > +1)^2}(\mu < g, E^A > - < E^A, x > -\lambda < g, E^A >) \\
&= \frac{< E^A, e_i >}{\lambda(< x, g > +1)} + \frac{< g, e_i >}{\lambda(< x, g > +1)^2}(- < E^A, x > + \frac{1 - \lambda}{\lambda\|g\|^2} < g, E^A >).
\end{aligned}$$

□

We also need the following Lemma 3.6, Lemma 3.7 and Lemma 3.8 to estimate the second eigenvalue $\lambda_2^{J_s}$ of the Jacobi operator J_s on M.

Lemma 3.6. *Let M be an n-dimensional compact hypersurface with constant scalar curvature $n(n-1)r$, $r \geq 1$, in $\mathbb{S}^{n+1}(1)$. Let f^A be the function given by (3.25), we have*

$$\sum_{A=1}^{n+2} \int_M (J_s f^A \cdot f^A) dv = \int_M \frac{n(n-1)H(1-\|g\|^2)}{(< x, g > +1)^2} dv - \int_M \{\frac{n(n-1)}{2}(2H-(n-2)H_3+nHH_2)\} dv. \tag{3.27}$$

Proof. By divergence theorem and Lemma 3.5 we have

$$\begin{aligned}
& -\sum_{A=1}^{n+2} \int_M (\Box f^A \cdot f^A) dv = \sum_{A=1}^{n+2} \int_M \sum_{i,j} (nH\delta_{ij} - h_{ij}) f_i^A f_j^A dv \\
& = \sum_{A=1}^{n+2} \int_M \sum_{i,j} (nH\delta_{ij} - h_{ij}) (\frac{< E^A, e_i >}{\lambda(< x, g > +1)} + \frac{< g, e_i >}{\lambda(< x, g > +1)^2} (- < E^A, x > + \frac{1-\lambda}{\lambda \|g\|^2} < g, E^A >)) \\
& \cdot (\frac{< E^A, e_j >}{\lambda(< x, g > +1)} + \frac{< g, e_j >}{\lambda(< x, g > +1)^2} (- < E^A, x > + \frac{1-\lambda}{\lambda \|g\|^2} < g, E^A >)) dv \\
& = \int_M \{\sum_{i,j} [nH\delta_{ij} - h_{ij}] [\frac{\delta_{ij}}{\lambda^2(< x, g > +1)^2} + \frac{< g, e_i >< g, e_j >}{\lambda^4 \|g\|^2 (< x, g > +1)^2} [2(1-\lambda)\lambda(< x, g > +1) \\
& + \lambda^2 \|g\|^2 - 2(1-\lambda)\lambda < x, g > + (1-\lambda)^2]]\} dv \\
& = \int_M \sum_{i,j} (nH\delta_{ij} - h_{ij}) \cdot \frac{\delta_{ij}}{\lambda^2(< x, g > +1)^2} dv \\
& = \int_M \frac{n(n-1)H(1-\|g\|^2)}{(< x, g > +1)^2} dv,
\end{aligned} \tag{3.28}$$

where we use the fact that $\sum\limits_{A=1}^{n+2} < E_A, X >< E_A, Y > = < X, Y >$ ($\forall\ X, Y \in \mathbb{R}^{n+2}$) in the third equality.

By Newton formula, we have

$$\begin{aligned}
f_3 &= n^3 H^3 + \frac{n(n-1)(n-2)}{2} H_3 - \frac{3n^2(n-1)}{2} HH_2, \\
S &= n^2 H^2 - n(n-1)H_2.
\end{aligned} \tag{3.29}$$

Thus J_s becomes

$$\begin{aligned}
J_s &= -\Box - \{n(n-1)H + nH(n^2H^2 - n(n-1)H_2) \\
&- (n^3H^3 + \frac{n(n-1)(n-2)}{2} H_3 - \frac{3n^2(n-1)}{2} HH_2)\} \\
&= -\Box - n(n-1)H - \frac{n^2(n-1)}{2} HH_2 + \frac{n(n-1)(n-2)}{2} H_3 \\
&= -\Box - \frac{n(n-1)}{2}(2H - (n-2)H_3 + nHH_2).
\end{aligned} \tag{3.30}$$

Then by using the fact that

$$\sum_{A=1}^{n+2} f^A \cdot f^A = \sum_{A=1}^{n+2} < E^A, F_g \circ x >< E^A, F_g \circ x > = < F_g \circ x, F_g \circ x > = 1, \tag{3.31}$$

we immediately get (3.27). $\Box$

For a fixed point $g \in B^{n+2}$, let

$$f = < x, g >, \ \tilde{g} = < e_{n+1}, g >, \ \rho = -\ln \lambda - \ln (1 + f), \tag{3.32}$$

where $\lambda = (1 - \|g\|^2)^{-1/2}$, x is the position vector and e_{n+1} is the unit normal vector. We have

$$e^{2\rho} = \frac{1}{\lambda^2(1+f)^2} = \frac{1 - \|g\|^2}{(< x, g > +1)^2}, \ \rho_i = \frac{-f_i}{1+f}, \ \rho_{ij} = \frac{-f_{ij}}{1+f} + \frac{f_i f_j}{(1+f)^2}. \tag{3.33}$$

Lemma 3.7. *Let $x : M \to \mathbb{S}^{n+1}(1)$ be an n-dimensional compact hypersurface with constant scalar curvature $n(n-1)r$, $r \geq 1$, in $\mathbb{S}^{n+1}(1)$. When $r = 1$, we assume moreover that $H_3 \neq 0$. Then we have $H \neq 0$, hence we can assume $H > 0$. Let ρ be the function defined by (3.32), we have*

$$\int_M \frac{H(1 - \|g\|^2)}{(< x, g > +1)^2} dv \leq \int_M (H + \frac{H_2^2}{H}) dv - \int_M [H\|\nabla \rho\|^2 - \frac{2}{n(n-1)} \sum_{i,j} (nH\delta_{ij} - h_{ij})\rho_i \rho_j] dv, \tag{3.34}$$

and the equality holds if and only if $H_2 + \frac{\tilde{g}H}{1+f} \equiv 0$ on M.

Proof. Under the hypothesis of the lemma, we can assume $H > 0$ (cf. [7] and [2]). We have

$$\sum_{i,j} (nH\delta_{ij} - h_{ij})\rho_i\rho_j = \sum_{i,j} (nH\delta_{ij} - h_{ij}) \frac{f_i f_j}{(1+f)^2} = \frac{nH\|\nabla f\|^2}{(1+f)^2} - \sum_{i,j} \frac{h_{ij} f_i f_j}{(1+f)^2}, \tag{3.35}$$

and

$$\begin{aligned} \Box \rho = \sum_{i,j} (nH\delta_{ij} - h_{ij})\rho_{ij} &= \sum_{i,j} (nH\delta_{ij} - h_{ij})(\frac{-f_{ij}}{1+f} + \frac{f_i f_j}{(1+f)^2}) \\ &= \frac{-\Delta f nH}{1+f} + \frac{nH\|\nabla f\|^2}{(1+f)^2} + \sum_{i,j} \frac{h_{ij} f_{ij}}{1+f} - \sum_{i,j} \frac{h_{ij} f_i f_j}{(1+f)^2}. \end{aligned} \tag{3.36}$$

From (3.33), (3.35) and (3.36) and by using Lemma 3.3, we have

$$\begin{aligned} &(\Box \rho - \sum_{i,j} (nH\delta_{ij} - h_{ij})\rho_i\rho_j) \cdot \frac{2}{n(n-1)} + \frac{H(1 - \|g\|^2)}{(1+f)^2} \\ &= (\frac{-\Delta f nH}{1+f} + \sum_{i,j} \frac{h_{ij} f_{ij}}{1+f}) \cdot \frac{2}{n(n-1)} + \frac{H(1 - \|g\|^2)}{(1+f)^2} \\ &= (\frac{-nH(nH\tilde{g} - nf)}{1+f} + \sum_{i,j} \frac{h_{ij}(\tilde{g}h_{ij} - f\delta_{ij})}{1+f}) \cdot \frac{2}{n(n-1)} + \frac{H(1 - \|g\|^2)}{(1+f)^2} \\ &= \frac{2Hf - 2H_2\tilde{g}}{1+f} + \frac{H(1 - f^2 - \sum_i f_i^2 - \tilde{g}^2)}{(1+f)^2} = H - \sum_i \frac{Hf_i^2}{(1+f)^2} - \frac{H\tilde{g}^2}{(1+f)^2} - \frac{2H_2\tilde{g}}{1+f} \\ &= H - \sum_i \frac{Hf_i^2}{(1+f)^2} + \frac{H_2^2}{H} - \frac{(H_2 + \frac{\tilde{g}H}{1+f})^2}{H} = H + \frac{H_2^2}{H} - H\|\nabla \rho\|^2 - \frac{(H_2 + \frac{\tilde{g}H}{1+f})^2}{H}, \end{aligned}$$

which immediately implies

$$\begin{aligned} &\int_M \frac{H(1 - \|g\|^2)}{(< x, g > +1)^2} dv \\ &= \int_M [H + \frac{H_2^2}{H} - H\|\nabla \rho\|^2 + \frac{2}{n(n-1)} \sum_{i,j} (nH\delta_{ij} - h_{ij})\rho_i\rho_j - \frac{(H_2 + \frac{\tilde{g}H}{1+f})^2}{H}] dv. \end{aligned} \tag{3.37}$$

Hence we get the inequality (3.34) and the equality holds if and only if $H_2 + \frac{\tilde{g}H}{1+f} \equiv 0$ on M. □

Lemma 3.8. *Let M be an n-dimensional compact hypersurface with constant scalar curvature $n(n-1)r,\ r \geq 1,$ in $\mathbb{S}^{n+1}(1),\ n \geq 5$. When $r = 1$, we assume moreover that $H_3 \neq 0$. Then we have $H \neq 0$, hence we can assume $H > 0$. We have*

$$\int_M [H\|\nabla\rho\|^2 - \frac{2}{n(n-1)} \sum_{i,j} (nH\delta_{ij} - h_{ij})\rho_i\rho_j] dv \geq 0. \tag{3.38}$$

Proof. Under the hypothesis of the lemma, we can assume $H > 0$ (cf. [7] and [2]). $\forall\, p \in M$, let $k_1, \ldots, k_n$ denote the principal curvatures of M at p, we choose an orthonormal basis such that $h_{ij} = \delta_{ij}k_i$. By Gauss equation (2.6), we have

$$n^2H^2 - \sum_i k_i^2 = n(n-1)(r-1) \geq 0, \tag{3.39}$$

which leads to

$$nH \geq |k_i|,\ \forall 1 \leq i \leq n. \tag{3.40}$$

As $n \geq 5$, we have $\frac{n(n-3)}{2}H \geq nH$, so we have

$$\begin{aligned}
&H\|\nabla\rho\|^2 - \frac{2}{n(n-1)} \sum_{i,j} (nH\delta_{ij} - h_{ij})\rho_i\rho_j \\
&= H\sum_i \rho_i^2 - \frac{2}{n(n-1)} \sum_{i,j} (nH\delta_{ij} - \delta_{ij}k_i)\rho_i\rho_j \\
&= H\sum_i \rho_i^2 - \sum_i \frac{2}{n(n-1)} (nH - k_i)\rho_i^2 \\
&= \frac{2}{n(n-1)} \sum_i \rho_i^2 (\frac{n(n-3)}{2} H + k_i) \geq \frac{2}{n(n-1)} \sum_i \rho_i^2 (nH - |k_i|) \geq 0.
\end{aligned}$$

Hence, we get $H\|\nabla\rho\|^2 - \frac{2}{n(n-1)} \sum_{i,j} (nH\delta_{ij} - h_{ij})\rho_i\rho_j \geq 0$ holds at every point of M, which immediately implies (3.38). $\square$

4 Proofs of Theorem 1.3 and Theorem 1.4

Proof of Theorem 1.3: Since $r > 1$, we have $\square$ is an elliptic operator and $H \neq 0$. Hence, we can assume $H > 0$ (see [7]). Let u be a first eigenfunction of J_s, we can assume u is positive on M, by Lemma 3.4 there exists $g \in B^{n+2}$ such that

$$\int_M u(F_g \circ x) dv = (0, \ldots, 0), \tag{4.1}$$

which implies that the functions $\{f^A,\ 1 \leq A \leq n+2\}$ given by (3.25) are perpendicular to the function u, i.e., $\int_M u \cdot f^A dv = 0,\ \forall 1 \leq A \leq n+2$. Then by using the min-max characterization of eigenvalues for elliptic operators, we have

$$\lambda_2^{J_s} \cdot \int_M (f^A \cdot f^A) dv \leq \int_M (J_s f^A \cdot f^A) dv,\ \forall\ 1 \leq A \leq n+2. \tag{4.2}$$

Summing up and using the fact that $\sum\limits_{A=1}^{n+2} f^A \cdot f^A = 1$ (see (3.31)), we obtain

$$\lambda_2^{J_s} \cdot Vol(M) \leq \sum_{A=1}^{n+2} \int_M (J_s f^A \cdot f^A) dv. \tag{4.3}$$

From Lemma 3.6 and (4.3) we have

$$\lambda_2^{J_s} \cdot Vol(M) \leq \int_M \frac{n(n-1)H(1-\|g\|^2)}{(<x,g>+1)^2}dv - \int_M \frac{n(n-1)}{2}(2H-(n-2)H_3+nHH_2)dv. \quad (4.4)$$

Then by (4.4), Lemma 3.7 and Lemma 3.8, we have

$$\begin{aligned}\lambda_2^{J_s} \cdot Vol(M) &\leq n(n-1)\cdot\int_M (H+\frac{H_2^2}{H})dv - \int_M \frac{n(n-1)}{2}(2H-(n-2)H_3+nHH_2)dv \\ &= n(n-1)\cdot\int_M (\frac{H_2^2}{H}+\frac{n-2}{2}H_3-\frac{nHH_2}{2})dv.\end{aligned} \quad (4.5)$$

From definition of H_2 and the Gauss equation (2.6) we have

$$H_2 = r-1 = \text{constant} > 0. \quad (4.6)$$

So we have $H_3 \leq \frac{H_2^2}{H}$ and $H_2 \leq H^2$ (see [10], p. 52) and hence

$$\begin{aligned}\lambda_2^{J_s} \cdot Vol(M) &\leq n(n-1)\cdot\int_M (\frac{H_2^2}{H}+\frac{n-2}{2}H_3-\frac{nHH_2}{2})dv \\ &\leq n(n-1)\cdot\int_M (\frac{H_2^2}{H}+\frac{n-2}{2}\frac{H_2^2}{H}-\frac{nHH_2}{2})dv \\ &= n(n-1)\cdot\int_M \frac{nH_2}{2}(\frac{H_2}{H}-H)dv \leq 0,\end{aligned} \quad (4.7)$$

therefore we get $\lambda_2^{J_s} \leq 0$.

When $\lambda_2^{J_s} = 0$, then all the inequalities become equalities. From (4.7) we have $H_2 = H^2$ on M, since H_2 is a positive constant, we get M is a totally umbilical and non-totally geodesic hypersurface with constant scalar curvature $n(n-1)r$. On the other hand, if M is a totally umbilical and non-totally geodesic hypersurface with constant scalar curvature $n(n-1)r$, from Example 3.1 in section 3, we know that $\lambda_2^{J_s} = 0$. □

Remark 4.1. We notice that from (4.7) we can get a more precise upper bound for $\lambda_2^{J_s}$, that is,

$$\begin{aligned}\lambda_2^{J_s} &\leq n(n-1)(\frac{H_2^2}{\min H}+\frac{n-2}{2}\max H_3-\frac{nH_2}{2}\min H) \\ &= n(n-1)(\frac{(r-1)^2}{\min H}+\frac{n-2}{2}\max H_3-\frac{n(r-1)}{2}\min H).\end{aligned} \quad (4.8)$$

Proof of Theorem 1.4: Since $r=1$, from (4.6) we have $H_2 = 0$. Since we assume that H_3 does not vanish on M, we have J_s is elliptic and the mean curvature H does not vanish on M(cf. Proposition 1.5 in [11]). Hence, we can assume $H > 0$. Thus $H_3 \leq \frac{H_2^2}{H} = 0$. Since we assume that $H_3 \neq 0$ on M, we get $H_3 < 0$. As Lemma 3.6, Lemma 3.7 and Lemma 3.8 hold for both the case $r > 1$ and the case $r = 1$, after an analogous argument with the proof of Theorem 1.3, we know that (4.1)-(4.5) still hold in this case, hence we have

$$\begin{aligned}\lambda_2^{J_s} \cdot Vol(M) &\leq n(n-1)\cdot\int_M (\frac{H_2^2}{H}+\frac{n-2}{2}H_3-\frac{nHH_2}{2})dv \\ &= \frac{n(n-1)(n-2)}{2}\cdot\int_M H_3 dv \\ &\leq \frac{n(n-1)(n-2)}{2}\max H_3 \cdot Vol(M) \\ &= -\frac{n(n-1)(n-2)}{2}\min|H_3|\cdot Vol(M).\end{aligned} \quad (4.9)$$

Hence, we get

$$\lambda_2^{J_s} \leq -\frac{n(n-1)(n-2)}{2} \min |H_3|. \tag{4.10}$$

When $\lambda_2^{J_s} = -\frac{n(n-1)(n-2)}{2} \min |H_3|$, the inequalities in (3.34), (4.2) and (4.9) become equalities. The equality in (4.9) holds implies that $H_3 = \text{constant} \neq 0$. Since $H_2 = 0$, the equalities in (3.34) holds implies that $\tilde{g} =< g, e_{n+1} >\equiv 0$ on M. We claim that g must be 0, otherwise, we have that M is a hypersphere (see Theorem 1 in [15]), hence M is totally umbilical, since $H_2 = 0$, we immediately get M is totally geodesic which is a contradiction with $H_3 \neq 0$. Hence we have $g \equiv 0$, from (3.25) we get $f^A =< E^A, F_g \circ x >=< E^A, x >$, which means $\{f^A,\ 1 \leq A \leq n+2\}$ are the position functions of $x : M \to \mathbb{S}^{n+1}(1)$. Since the equality in (4.2) holds, it follows that the position functions $\{f^A =< E^A, x >,\ 1 \leq A \leq n+2\}$ must be the second eigenfunctions of J_s corresponding to $\lambda_2^{J_s}$.

On the other hand, if we assume that $H_3 = \text{constant} \neq 0$ and the position functions $\{\tilde{f}^A =< E^A, x >,\ 1 \leq A \leq n+2\}$ are the second eigenfunctions of J_s corresponding to $\lambda_2^{J_s}$. Since $H_3 \neq 0$, we have $H \neq 0$. Hence, we can assume $H > 0$, $H_3 < 0$ (cf. Proposition 1.5 in [11]).

Since $H_2 = 0$, by using (1.1) and (3.20), we get

$$\Box \tilde{f}^A = n(n-1)H_2 < E^A, e_{n+1} > -n(n-1)H\tilde{f}^A = -n(n-1)H\tilde{f}^A,\ \forall\, 1 \leq A \leq n+2,$$

then from (1.2) and (3.29) we have

$$\begin{aligned}
J_s \tilde{f}^A &= n(n-1)H\tilde{f}^A - \{n(n-1)H + nHS - f_3\}\tilde{f}^A \\
&= (f_3 - nHS)\tilde{f}^A \\
&= \{(n^3H^3 + \frac{n(n-1)(n-2)}{2}H_3 - \frac{3n^2(n-1)}{2}HH_2) - (n^3H^3 - n^2(n-1)HH_2)\}\tilde{f}^A \\
&= \frac{n(n-1)(n-2)}{2}H_3\tilde{f}^A,\ \forall\, 1 \leq A \leq n+2,
\end{aligned}$$

hence we get $\lambda_2^{J_s} = \frac{n(n-1)(n-2)}{2}H_3 = -\frac{n(n-1)(n-2)}{2}\min |H_3|$.

In particular, when M is the Riemannian product $\mathbb{S}^m(c) \times \mathbb{S}^{n-m}(\sqrt{1-c^2})$, $1 \leq m \leq n-2$ with $c = \sqrt{\frac{(n-1)m + \sqrt{(n-1)m(n-m)}}{n(n-1)}}$, from Example 3.3 in section 3, we know that the equality in (4.10) is attained. $\square$

Remark 4.2. Since Lemma 3.8 does not hold when $n = 3$ and $n = 4$, we can not prove Theorem 1.3 and Theorem 1.4 by our technique in $n = 3$ and $n = 4$. So it is an interesting problem to study the estimate for the second eigenvalue of the Jacobi oeprator J_s of the hypersurface $x : M^n \to \mathbb{S}^{n+1}(1)$ when $n = 3$ and $n = 4$.

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